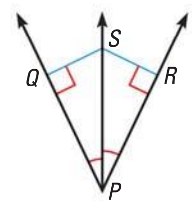
5.3 Use Angle Bisectors of Triangles

Before You used angle bisectors to find angle relationships.
Now You will use angle bisectors to find distance relationships.
Why? So you can apply geometry in sports, as in Example 2.

Key Vocabulary
- **incenter**
- **angle bisector,** *p. 28*
- **distance from a point to a line,** *p. 192*

Remember that an *angle bisector* is a ray that divides an angle into two congruent adjacent angles. Remember also that the *distance from a point to a line* is the length of the perpendicular segment from the point to the line.

So, in the diagram, $\overrightarrow{PS}$ is the bisector of $\angle QPR$ and the distance from S to $\overrightarrow{PQ}$ is SQ, where $\overline{SQ} \perp \overrightarrow{PQ}$.

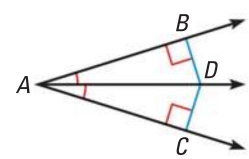


THEOREMS — For Your Notebook

THEOREM 5.5 Angle Bisector Theorem

If a point is on the bisector of an angle, then it is equidistant from the two sides of the angle.

If $\overrightarrow{AD}$ bisects $\angle BAC$ and $\overline{DB} \perp \overrightarrow{AB}$ and $\overline{DC} \perp \overrightarrow{AC}$, then $DB = DC$.

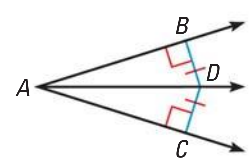


Proof: Ex. 34, p. 315

THEOREM 5.6 Converse of the Angle Bisector Theorem

If a point is in the interior of an angle and is equidistant from the sides of the angle, then it lies on the bisector of the angle.

If $\overline{DB} \perp \overrightarrow{AB}$ and $\overline{DC} \perp \overrightarrow{AC}$ and $DB = DC$, then $\overrightarrow{AD}$ bisects $\angle BAC$.

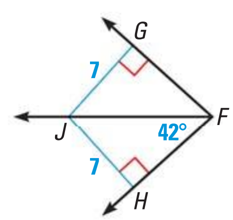


Proof: Ex. 35, p. 315

REVIEW DISTANCE
In Geometry, *distance* means the *shortest* length between two objects.

EXAMPLE 1 Use the Angle Bisector Theorems

Find the measure of $\angle GFJ$.

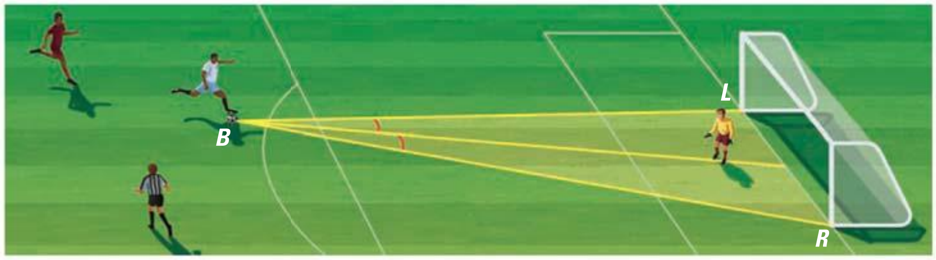


Solution

Because $\overline{JG} \perp \overrightarrow{FG}$ and $\overline{JH} \perp \overrightarrow{FH}$ and $JG = JH = 7$, $\overrightarrow{FJ}$ bisects $\angle GFH$ by the Converse of the Angle Bisector Theorem. So, $m\angle GFJ = m\angle HFJ = 42°$.

EXAMPLE 2 Solve a real-world problem

SOCCER A soccer goalie's position relative to the ball and goalposts forms congruent angles, as shown. Will the goalie have to move farther to block a shot toward the right goalpost R or the left goalpost L?



Solution

The congruent angles tell you that the goalie is on the bisector of $\angle LBR$. By the Angle Bisector Theorem, the goalie is equidistant from $\overrightarrow{BR}$ and $\overrightarrow{BL}$.

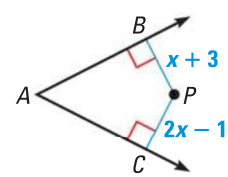
▶ So, the goalie must move the same distance to block either shot.

EXAMPLE 3 Use algebra to solve a problem

xy ALGEBRA For what value of x does P lie on the bisector of $\angle A$?

Solution

From the Converse of the Angle Bisector Theorem, you know that P lies on the bisector of $\angle A$ if P is equidistant from the sides of $\angle A$, so when $BP = CP$.



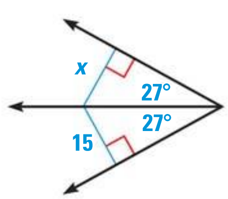
$BP = CP$	**Set segment lengths equal.**
$x + 3 = 2x - 1$	**Substitute expressions for segment lengths.**
$4 = x$	**Solve for x.**

▶ Point P lies on the bisector of $\angle A$ when $x = 4$.

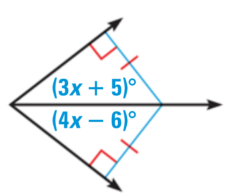
✓ GUIDED PRACTICE for Examples 1, 2, and 3

In Exercises 1–3, find the value of x.

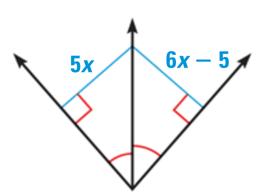
1.



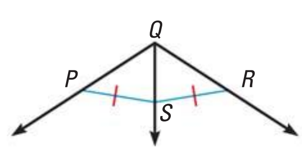
2.



3.



4. Do you have enough information to conclude that $\overrightarrow{QS}$ bisects $\angle PQR$? Explain.



READ VOCABULARY

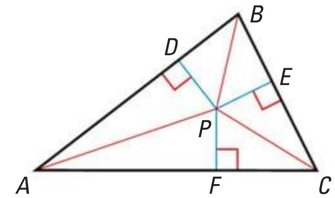
An *angle bisector of a triangle* is the bisector of an interior angle of the triangle.

THEOREM *For Your Notebook*

THEOREM 5.7 Concurrency of Angle Bisectors of a Triangle

The angle bisectors of a triangle intersect at a point that is equidistant from the sides of the triangle.

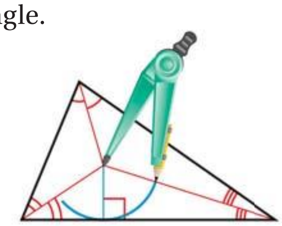
If $\overline{AP}$, $\overline{BP}$, and $\overline{CP}$ are angle bisectors of $\triangle ABC$, then $PD = PE = PF$.

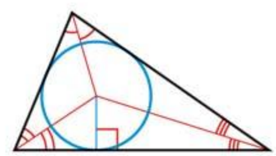


Proof: Ex. 36, p. 316

The point of concurrency of the three angle bisectors of a triangle is called the **incenter** of the triangle. The incenter always lies inside the triangle.

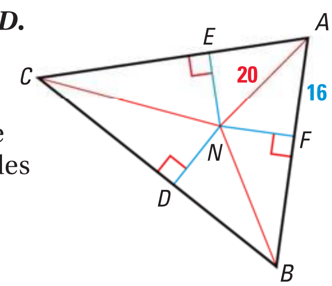
Because the incenter P is equidistant from the three sides of the triangle, a circle drawn using P as the center and the distance to one side as the radius will just touch the other two sides. The circle is said to be *inscribed* within the triangle.





EXAMPLE 4 Use the concurrency of angle bisectors

In the diagram, N is the incenter of $\triangle ABC$. Find ND.



Solution

REVIEW QUADRATIC EQUATIONS

For help with solving a quadratic equation by taking square roots, see page 882. Use only the positive square root when finding a distance, as in Example 4.

By the Concurrency of Angle Bisectors of a Triangle Theorem, the incenter N is equidistant from the sides of $\triangle ABC$. So, to find ND, you can find NF in $\triangle NAF$. Use the Pythagorean Theorem stated on page 18.

$c^2 = a^2 + b^2$	Pythagorean Theorem
$20^2 = NF^2 + 16^2$	Substitute known values.
$400 = NF^2 + 256$	Multiply.
$144 = NF^2$	Subtract 256 from each side.
$12 = NF$	Take the positive square root of each side.

▶ Because $NF = ND$, $ND = 12$.

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✓ GUIDED PRACTICE for Example 4

5. WHAT IF? In Example 4, suppose you are not given AF or AN, but you are given that $BF = 12$ and $BN = 13$. Find ND.

5.3 EXERCISES

HOMEWORK KEY

○ = **WORKED-OUT SOLUTIONS** on p. WS1 for Exs. 7, 15, and 29

★ = **STANDARDIZED TEST PRACTICE** Exs. 2, 18, 23, 30, and 31

SKILL PRACTICE

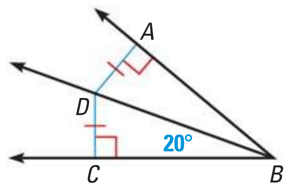
1. **VOCABULARY** Copy and complete: Point C is in the interior of $\angle ABD$. If $\angle ABC$ and $\angle DBC$ are congruent, then $\overrightarrow{BC}$ is the __?__ of $\angle ABD$.

2. ★ **WRITING** How are perpendicular bisectors and angle bisectors of a triangle different? How are they alike?

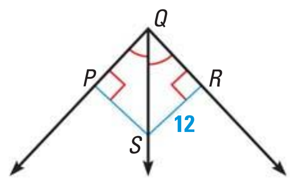
EXAMPLE 1 on p. 310 for Exs. 3–5

FINDING MEASURES Use the information in the diagram to find the measure.

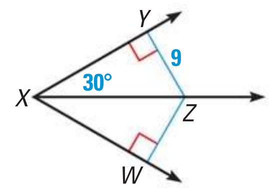
3. Find $m\angle ABD$.



4. Find PS.



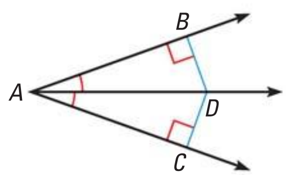
5. $m\angle YXW = 60°$. Find WZ.



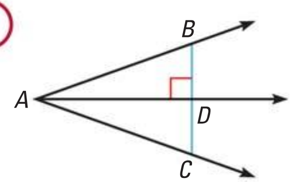
EXAMPLE 2 on p. 311 for Exs. 6–11

ANGLE BISECTOR THEOREM Is $DB = DC$? *Explain.*

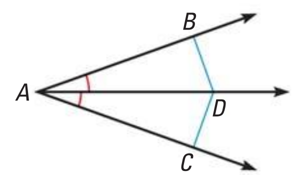
6.



7.

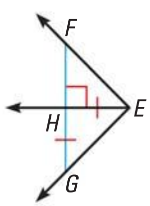


8.

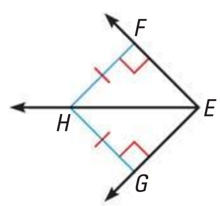


REASONING Can you conclude that $\overrightarrow{EH}$ bisects $\angle FEG$? *Explain.*

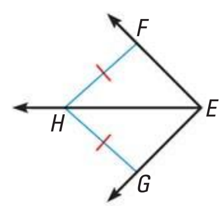
9.



10.



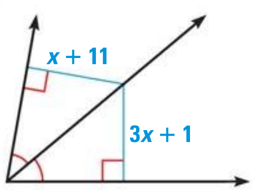
11.



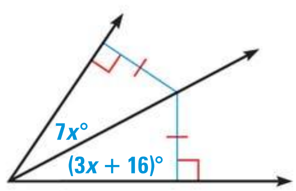
EXAMPLE 3 on p. 311 for Exs. 12–18

xy **ALGEBRA Find the value of x.**

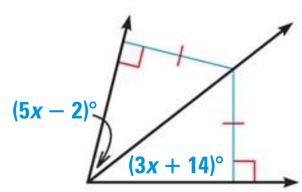
12.



13.

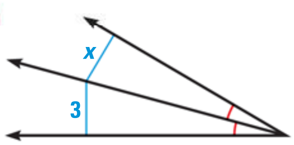


14.

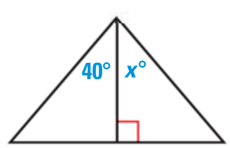


RECOGNIZING MISSING INFORMATION Can you find the value of x? *Explain.*

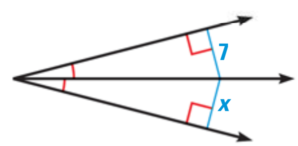
15.



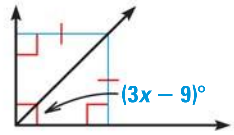
16.



17.



18. ★ **MULTIPLE CHOICE** What is the value of x in the diagram?



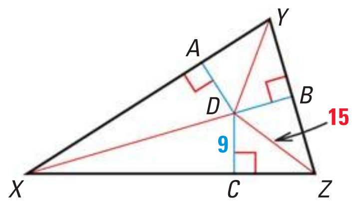
Ⓐ 13 Ⓑ 18

Ⓒ 33 Ⓓ Not enough information

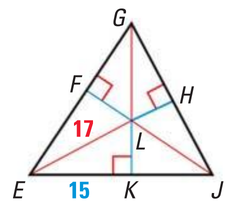
EXAMPLE 4 on p. 312 for Exs. 19–22

USING INCENTERS **Find the indicated measure.**

19. Point D is the incenter of $\triangle XYZ$. Find DB.

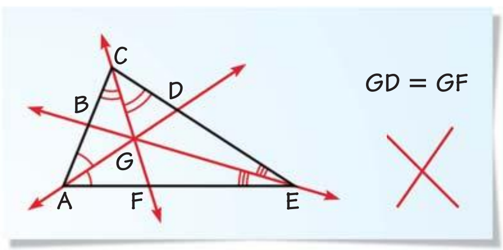


20. Point L is the incenter of $\triangle EGJ$. Find HL.

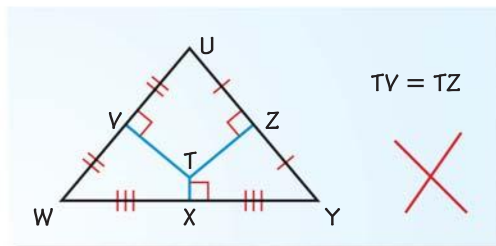


ERROR ANALYSIS ***Describe*** **the error in reasoning. Then state a correct conclusion about distances that can be deduced from the diagram.**

21.



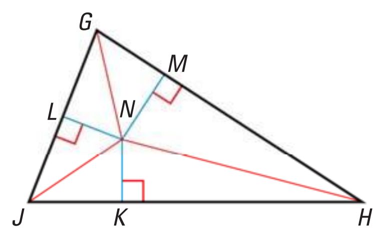
22.



23. ★ **MULTIPLE CHOICE** In the diagram, N is the incenter of $\triangle GHJ$. Which statement cannot be deduced from the given information?

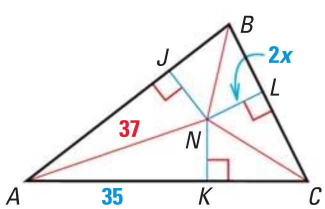
Ⓐ $\overline{NM} \cong \overline{NK}$ Ⓑ $\overline{NL} \cong \overline{NM}$

Ⓒ $\overline{NG} \cong \overline{NJ}$ Ⓓ $\overline{HK} \cong \overline{HM}$

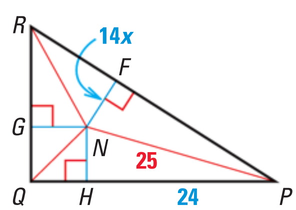


ALGEBRA **Find the value of x that makes N the incenter of the triangle.**

24.

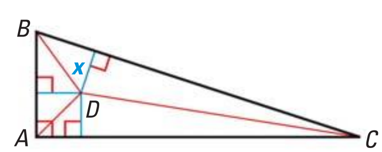


25.



26. **CONSTRUCTION** Use a compass and a straightedge to draw $\triangle ABC$ with incenter D. Label the angle bisectors and the perpendicular segments from D to each of the sides of $\triangle ABC$. Measure each segment. What do you notice? What theorem have you verified for your $\triangle ABC$?

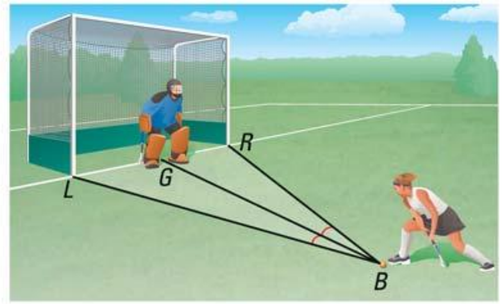
27. **CHALLENGE** Point D is the incenter of $\triangle ABC$. Write an expression for the length x in terms of the three side lengths AB, AC, and BC.



PROBLEM SOLVING

EXAMPLE 2
on p. 311
for Ex. 28

28. FIELD HOCKEY In a field hockey game, the goalkeeper is at point G and a player from the opposing team hits the ball from point B. The goal extends from left goalpost L to right goalpost R. Will the goalkeeper have to move farther to keep the ball from hitting L or R? *Explain*.



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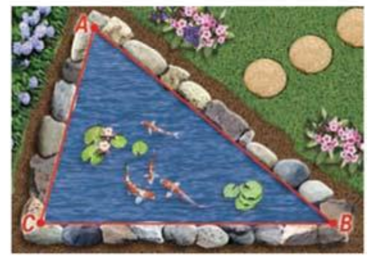
29. KOI POND You are constructing a fountain in a triangular koi pond. You want the fountain to be the same distance from each edge of the pond. Where should you build the fountain? *Explain* your reasoning. Use a sketch to support your answer.



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30. ★ SHORT RESPONSE What congruence postulate or theorem would you use to prove the Angle Bisector Theorem? to prove the Converse of the Angle Bisector Theorem? Use diagrams to show your reasoning.

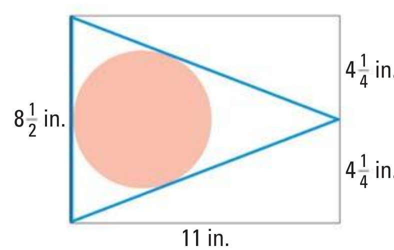
31. ★ EXTENDED RESPONSE Suppose you are given a triangle and are asked to draw all of its perpendicular bisectors and angle bisectors.

a. For what type of triangle would you need the fewest segments? What is the minimum number of segments you would need? *Explain*.

b. For what type of triangle would you need the most segments? What is the maximum number of segments you would need? *Explain*.

CHOOSING A METHOD In Exercises 32 and 33, tell whether you would use *perpendicular bisectors* or *angle bisectors*. Then solve the problem.

32. BANNER To make a banner, you will cut a triangle from an $8\frac{1}{2}$ inch by 11 inch sheet of white paper and paste a red circle onto it as shown. The circle should just touch each side of the triangle. Use a model to decide whether the circle's radius should be *more* or *less* than $2\frac{1}{2}$ inches. Can you cut the circle from a 5 inch by 5 inch red square? *Explain*.



33. CAMP A map of a camp shows a pool at (10, 20), a nature center at (16, 2), and a tennis court at (2, 4). A new circular walking path will connect the three locations. Graph the points and find the approximate center of the circle. Estimate the radius of the circle if each unit on the grid represents 10 yards. Then use the formula $C = 2\pi r$ to estimate the length of the path.

PROVING THEOREMS 5.5 AND 5.6 Use Exercise 30 to prove the theorem.

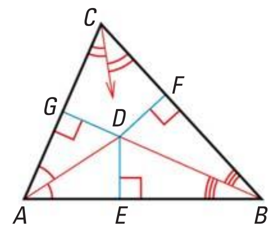
34. Angle Bisector Theorem

35. Converse of the Angle Bisector Theorem

36. **PROVING THEOREM 5.7** Write a proof of the Concurrency of Angle Bisectors of a Triangle Theorem.

GIVEN ▶ $\triangle ABC$, $\overline{AD}$ bisects $\angle CAB$, $\overline{BD}$ bisects $\angle CBA$, $\overline{DE} \perp \overline{AB}$, $\overline{DF} \perp \overline{BC}$, $\overline{DG} \perp \overline{CA}$

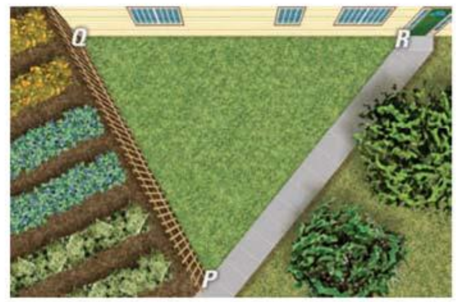
PROVE ▶ The angle bisectors intersect at D, which is equidistant from $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$.



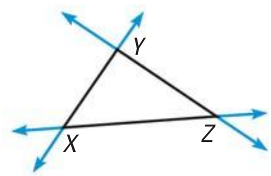
37. **CELEBRATION** You are planning a graduation party in the triangular courtyard shown. You want to fit as large a circular tent as possible on the site without extending into the walkway.

a. Copy the triangle and show how to place the tent so that it just touches each edge. Then *explain* how you can be sure that there is no place you could fit a larger tent on the site. Use sketches to support your answer.

b. Suppose you want to fit as large a tent as possible while leaving at least one foot of space around the tent. Would you put the center of the tent in the same place as you did in part (a)? *Justify* your answer.



38. **CHALLENGE** You have seen that there is a point inside any triangle that is equidistant from the three sides of the triangle. Prove that if you extend the sides of the triangle to form lines, you can find three points outside the triangle, each of which is equidistant from those three lines.



MIXED REVIEW

PREVIEW
Prepare for Lesson 5.4 in Exs. 39–41.

Find the length of $\overline{AB}$ and the coordinates of the midpoint of $\overline{AB}$. *(p. 15)*

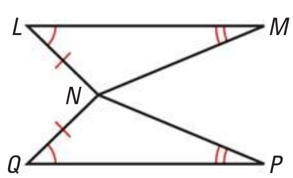
39. $A(-2, 2)$, $B(-10, 2)$

40. $A(0, 6)$, $B(5, 8)$

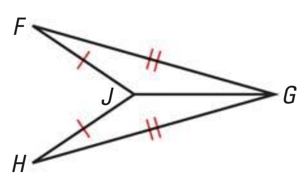
41. $A(-1, -3)$, $B(7, -5)$

Explain **how to prove the given statement.** *(p. 256)*

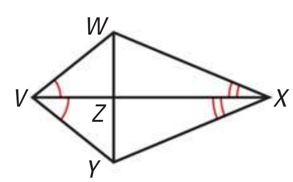
42. $\angle QNP \cong \angle LNM$



43. $\overline{JG}$ bisects $\angle FGH$.

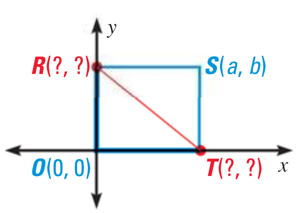


44. $\triangle ZWX \cong \triangle ZYX$

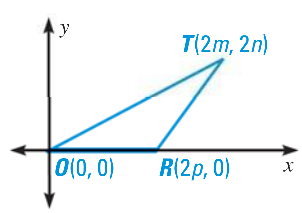


Find the coordinates of the red points in the figure if necessary. Then find *OR* and the coordinates of the midpoint *M* of $\overline{RT}$. *(p. 295)*

45.



46.



47.